The Twin Paradox

1, Introduction

Two twins, Art and Beth, have an adventure. One day, Beth takes off in a spaceship and flies at great speed for a long time. Eventually she turns around and flies home at the same speed, only to find Art has aged much more than she. Such curious facts are often summed up with the catch-phrase, “moving clocks go slow.” But things are still more curious when you consider that motion is relative.

If Beth flies by Art in her ship, he will see her clock tick slowly. The watch on his wrist shows it took 2 seconds for her to travel from one place to another, but he can see that the clock on her ship only moved forward 1 second during this travel. From Beth’s perspective, she is just sitting in her ship and Art is flying by her. And she can see that Art’s watch is ticking slowly compared to her clock, measuring off only half a second as her clock moves a full second.

This is so-called the twin paradox. How can Beth’s clock move slower than Art’s and yet Art’s clock move slower than Beth’s? Beth’s trip to a great distance and back again is the traditional way to highlight the problem. If both have slow clocks, that is, age slowly, which twin is younger when Beth comes home?

We already mentioned that it is Beth who is younger, and the reason for that we will come to presently. But there is a key part of the paradox which doesn’t require a round trip. It is contained in a simple flyby. How can Beth’s clock move slower then Art’s and yet Art’s move slower than Beth’s?

The answer is that both happen because the clocks are moving not just in space but also in time, so that moving clocks at different places exist in different times.

2. Relative motion

To see how this works, let’s look at a simple flyby step by step, as Beth flies her ship at great speed from west to east past Art on the Earth. To make the details clear, we are going to add some clocks. Art has positioned space stations at equal intervals both to the east and to the west of the Earth, and labeled them E1, E2, and W1, W2, and so on. The stations are all motionless with respect to the Earth, and each has its own clock, which keeps accurate Earth time.

Similarly, Beth has deployed a series of drones at equal intervals behind and in front of her ship. Since she is flying from west to east, these drones are west and east of the ship and labeled like the stations, E1, E2, W1, etc. All the drones are moving at the same speed as the ship, and their clocks keep accurate ship time. The interval between drones is exactly the same as the interval between stations. Both Beth and Art will keep a careful record of the time showing on their clocks at each step of the flyby.

For the moment we will ignore how Beth came to be traveling at great speed by the Earth. For our purposes she (or at least her ship) can have been moving at this speed forever. Also, it doesn’t matter if Art and Beth are different ages. We are only interested in how they age during the flyby.

As Beth passes Art on the Earth, both siblings mark the time on their clocks exactly, and each will use this time as a reference point, subtracting it from all of their recorded times. This way, both Art's and Beth's clocks will officially be at zero time the one instant the two are in the same place.

In Figure 1 are five pictures which show the flyby from Art's point of view. The times on his clock are -1, -0.5, 0, +0.5 and +1 second, respectively. The upper half of each picture is Beth in her ship, along with her drones. Art and the two nearest of his stations are shown in the lower half of each picture. Red labels are used for drones and blue for stations.

Figure 2 gives five pictures from Beth's point of view. She and her drones watch as Art and his stations fly by her moving east to west. The same sequence of times is used for the ship clock, -1, -0.5, 0, +0.5 and +1 second. Again Beth is in the upper part of each picture, with her drones, and Art in the lower, with his stations.

In Figure 1, then, one thing that is immediately obvious is that all the moving clocks are shortened, as indeed are the distances. In fact the distances have been cut in half. For example, the distance from the ship to drone E1 is normally (no relative motion) the same as the distance from Art to station E1, but as Art can plainly see this is no longer so. It is also clear that the ship clock is moving slowly. From the top picture to the bottom, two seconds pass for Art, but only one for Beth. The ship clock in particular goes from -0.5 to +0.5 seconds, which is 1 second elapsed. And although all the drone clocks likewise increase by 1 second, it is a curious fact that at any one time on Art's clock, the drone clocks all read differently.

In Figure 2, the situation is completely analogous to that of Figure 1. Beth sees the moving clocks on the Earth and the stations are all shortened and ticking slowly. Again both distances and times are cut in half.

As far as it goes, then, this creates a satisfying picture. Everything is symmetric. Motion is purely relative, and either observer can witness time dilation and length contraction in the other's clocks. But something seems to be off. For example, in the top picture of Figure 1, Beth's clock reads -0.5 while Art's clock reads -1. Yet in the second picture of Figure 2, Beth's clock also reads -0.5, while Art's clock says not -1, but -0.25.

This is because the two pictures are seen from different frames, and because Beth and Art are in different places. Only when things are in the same place are the times always unchanged. In this example, Beth's clock is -0.5 when she is at Station W1 and that clock reads -1, and this true in both Figure 1 (top picture) and Figure 2 (2nd picture), that is, from the perspective of either observer.

In fact, from the station clock where she is at any particular time T (her clock), she can tell what time Art and the stations, from their point of view, will see her clock read T. For example, in Figure 2, picture 4, the station E1 clock tells Beth that Art and his stations will see her clock at 0.5 when they are at 1.0, as shown in the bottom picture of Figure 1. From this we know, looking at the top picture of Figure 2, that Art's clock, and the station clocks, were at -2 seconds (W2) when they saw Beth's clock at -1, even though this is not shown explicitly in Figure 1.

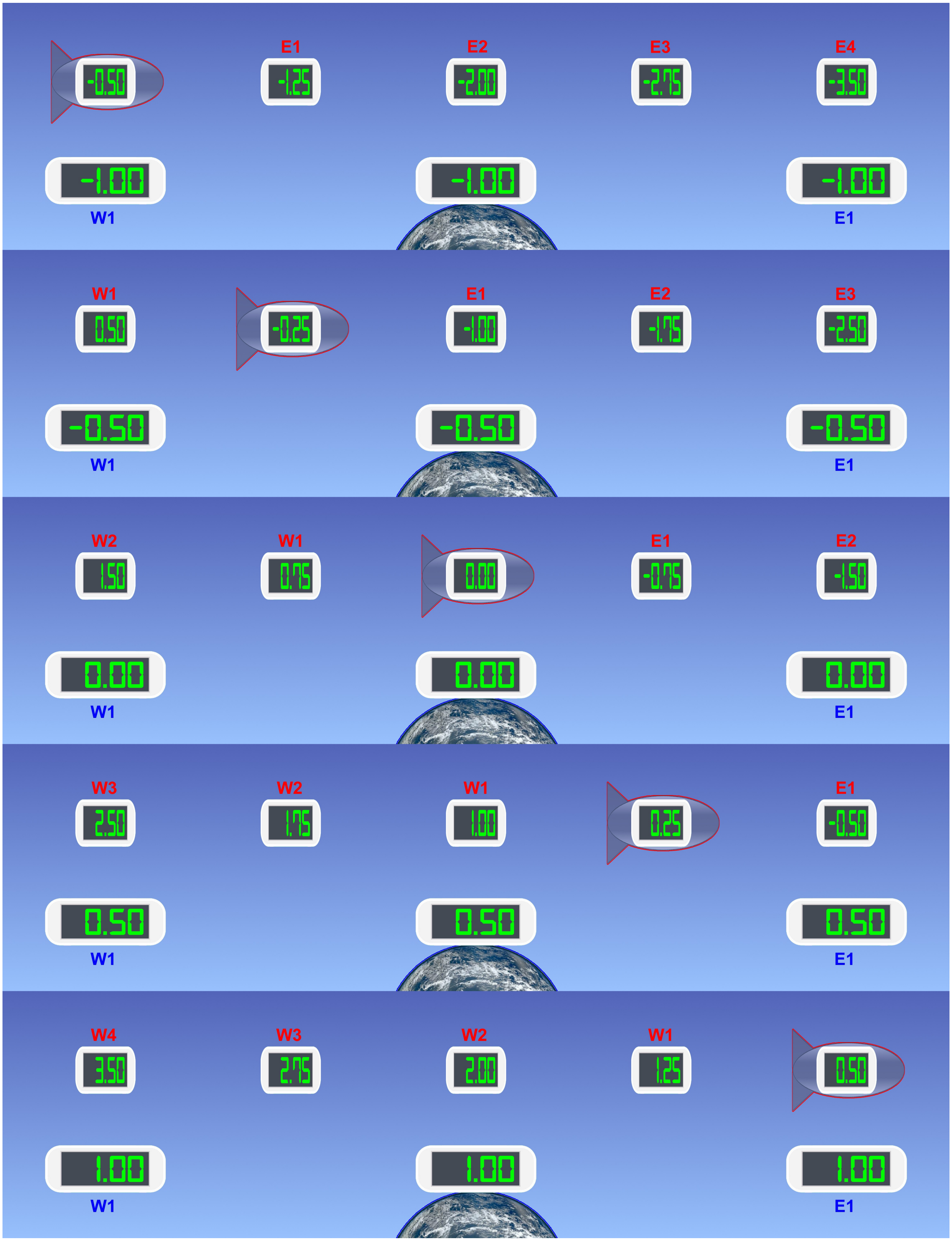

Figure 1. The flyby from Art's point of view on Earth. See the text for details.

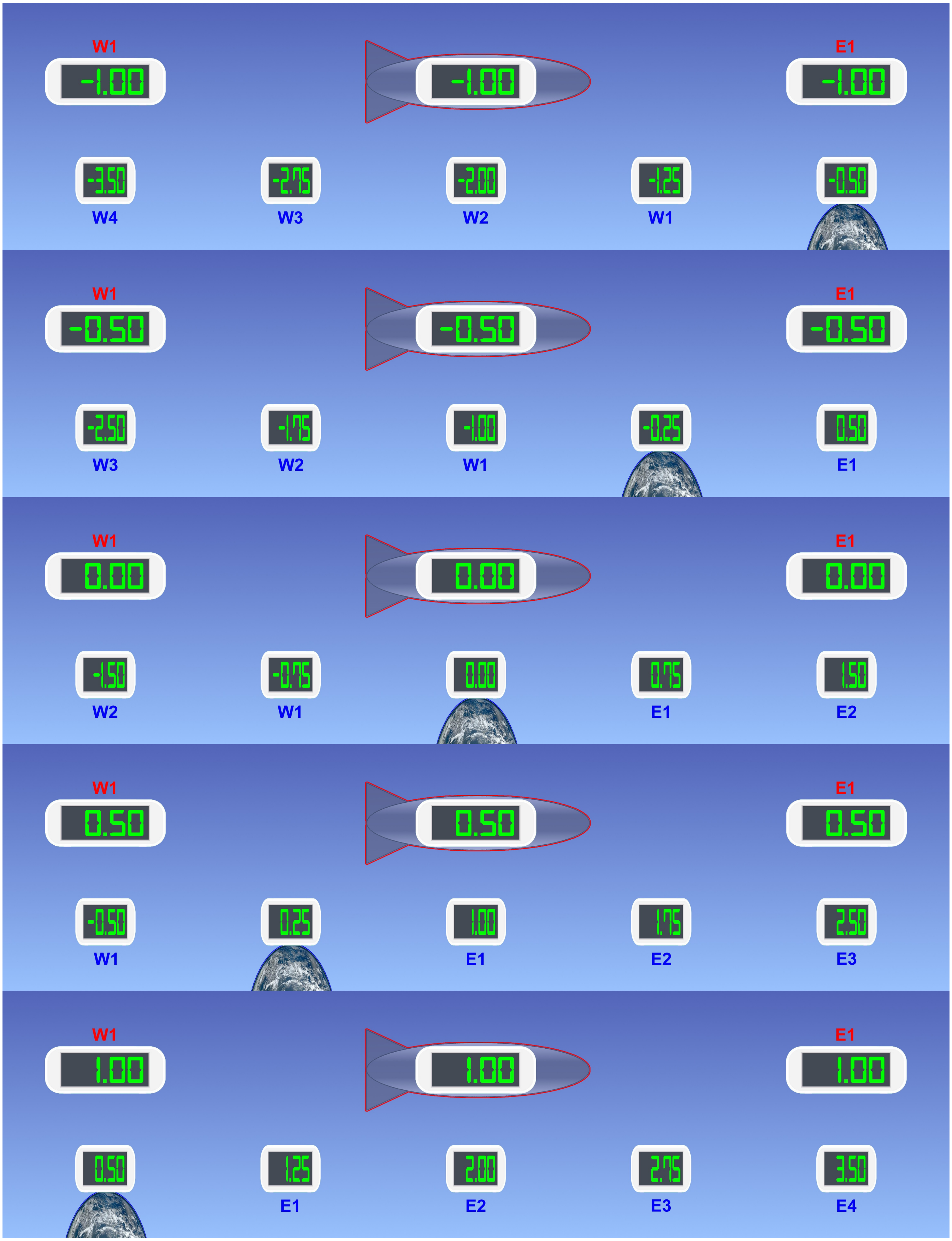

Figure 2. The flyby from Beth's point of view on the ship. See the text for details.

The reason the moving clocks read differently is that they exist at different times. Events which are simultaneous to Art and occur a distance D apart occur (at the same positions) for Beth at different times, with a difference proportional to D. Events at the downstream location (Drone E2, say, compared to E1) occur earlier for Beth. For example, if Beth flashes her laser pointer when her clock says 0.25, and 1.5 seconds later Drone W2 blinks a navigation light, Art will see these events happen at the same time, namely 0.5 on his clock (Figure 1, picture 4).

One other effect is worth noting. As long as either twin focuses on the other, each will see the other's clock moving at half speed. On the other hand, if Art looks not at the ship clock, but at the drone clocks *as they pass his position* he will see the time advancing at double the speed of his clock. In Figure 1, the time on drone E2 is -2 in the top picture and the time on W2 is +2 in the bottom. While Art's clock has moved forward 2 seconds, the "Beth" clocks at the Earth show an increase of 4 seconds. The same is true for Beth looking out her porthole at the passing stations.

Another way to put it is that if you watch a fixed point in the moving system, you see time moving slow, but if you watch the moving system at a fixed point in your own, you see the time moving fast.

So this is the situation. As long as Beth keeps flying, as long as the relative motion continues, each twin will continue to see the other's clock running at half speed, and each will continue to see the moving time "outside their window" running at double speed. This can go on indefinitely. As to which twin is younger, each twin sees the other as aging slowly, with the age difference increasing indefinitely.

However, if Beth stops, it is a different story. Let's say that in the bottom picture of Figure 2, which is shown from Art's perspective in Figure 3, she suddenly stops. She finds herself at Station E2, having aged 1 second since she passed Art. Art, by contrast, has aged 2 seconds, as evidenced by his clock. So Beth is younger, or more accurately, has aged less than Art, only half as much in fact. If it had been not 1 second but, say, 1 decade, it would be much more dramatic, but the principle is exactly the same.

Figure 3. The ship at Station E2 (bottom picture of Figure 2) from Art's perspective.

But now suppose, not that Beth stops, but rather that Art is suddenly moving at the same speed as Beth, starting when he is just even with drone W2. This point is shown in the bottom picture of Figure 1 and, from Beth's perspective, in Figure 4. Art finds himself moving with drone W2, having aged 1 second to Beth's 2, i.e., Art has aged only half as much as Beth. The reason is that it is he who has changed his speed, whereas in the case of Beth stopping, it is she. It is for this same reason that in the classic round trip we talked about in the Introduction, it is the traveling twin who is the younger. In that case she changes her speed not just by stopping, but by actually reversing herself.

Figure 4. The Earth at drone W2 (bottom picture of Figure 1) from Beth's perspective.

In closing, we need to be clear that in all of this when we say "see," we are talking about when and where things happen as opposed to when they are actually observed. When we say, for example, that Art sees Beth's clock reading 0.5 as she passes station E1 at 1 second on his clock, we are pretending that he will see things instantly, without any time lag due to the travel time of the light from E1 to Earth. In the real world, the light travel time is a complication, but as we will see in the next section, it does not affect the timing and placement of events as described here.

3. Life on the move

Before moving on to the round trip, let's consider some of the practical implications of the "paradox" of the flyby, the mutually flagging clocks. First we should point out that by choosing a dilation/contraction ratio of 2 to 1, we have set Beth's speed v to 0.866 c, where c is the speed of light, and the separation of the stations/drones to 0.866 light-seconds (~260,000 km). In order to make it more interesting, let us now move the stations/drones out to much greater distances, say 26 light-days between each (~674 billion km). This choice results in 30 days travel time between stations or drones (as observed by Art or Beth, respectively).

On day 15 of Beth's calendar, she passes station E1 and sends a message to Art, "Hey Art, day 15, just passed E1." This event, Beth at E1, happens on day 30 of Art's calendar, but already on day 15 he has passed drone W1, where he sent Beth a message, "Hey Beth, day 15, W1 just passed overhead." At that time, Beth was half-way to E1.

Now Beth's message has a travel time of 26 days, so Art gets the message on day 56. (Knowing the travel time, Art concludes that Beth's calendar was 15 days behind his when she transmitted.) He sends back, "Hello, Beth, that's great. Day 56 here and we copy your E1 milestone," for at the same time he received her message, the light generally that left E1 when Beth passed also arrived, and he could see in his telescope that she had indeed made it to E1. By this time Beth has moved on another 22.5 light-days, by Art's measure.

As for Art's message from W1, that catches up with Beth at a distance d such that $d/c = (d - d_0)/v$, where $d_0 =$ 13 light-days, Beth's location when the message was sent. This comes to $d =$ 97 light-days, so Beth is almost to Station E4 and the time on Art's calendar is now day 112 (= 97/0.866).

Art's reply to Beth's E1 message then intercepts Beth at a distance d such that $d/c = (d - d_0)/v$, where $d_0 =$ 26 + 22.5 = 48.5 light-days, and this is $d =$ 362 light-days, right at Station E14. The time is day 418 on Art's calendar, and day 209 on Beth's.

From Beth's perspective, she reaches E1 on day 15 when Earth is halfway between her and drone W1, 13 light-days behind her. She sends her message and it makes it to Earth when Earth is 97 light-days from her, almost to W4. It is day 112 for her. Then she receives a message from Art 97 days later, on day 209, replying to her message and telling her he got it on day 56 of his calendar.

However, well before her E1 message gets to Art, on day 56, she receives his message from W1. (Knowing the message travel time, she concludes that Art's calendar was 15 days behind hers when he transmitted.) Like Art, she checks her telescope and sends back, "Copy that, Art, W1 at Earth." This reply gets to Art on Beth's day 418 as drone W14 passes by the Earth.

To recap from Art's point of view, he sends a message to Beth on day 15 when W1 passes by. Beth reaches Station E1, 26 light-days away, on day 30. He receives her message from there on day 56 and replies. Then on day 112 he gets her reply to his W1 message (sent on Beth's day 56). His own reply (to the E1 missive) makes it to the ship on day 418, when Beth is 362 light-days away at station E14.

To recap from Beth's point of view, Art has made it half-way to drone W1 on day 15 when she passes station E1, 13 light-days from Earth. She sends a message that gets to Art on day 112, when he is about 6 light-days short of W4, and long after she got his W1 message on day 56. She gets his E1 reply on day 209, when Art is at W7, and her reply to Art's W1 message arrives at the Earth on day 418, as W14 passes overhead there.

Thus the lives of the mutually traveling twins are perfectly coherent, and the events for each are perfectly symmetric. The light travel-time is a complication for observation, but it meshes seamlessly with the narrative. Each twin steadfastly sees the other aging at half the rate of their own aging. *** needs a graphic***

4. The journey

Now we will consider Beth's round trip from Earth to far away and back. To simplify the discussion, let us generalize the chain of stations into a coordinate axis. We will call the distance along the axis x and locate the Earth, and Art, at $x = 0$. The time on the clocks keeping Earth time will be t. Similarly, we have a second coordinate axis labeled by x' for Beth and the drones. This axis will move east with Beth so that she is always at $x' = 0$, and the time on the clocks will be t'. As before $t = t' = 0$ at $x = x' = 0$, when Art and Beth are together at the start.

In thinking ahead about Beth's return trip, we also define a third axis x'', with a double-prime, or d-prime, which moves to the west at Beth's speed v. Such coordinate systems are often called frames of reference, or just frames, and we will use S for the Earth frame, S' for the eastbound frame and S'' for the westbound frame. As with the other two, we set $x'' = 0$ and $t'' = 0$ on Earth at the start, when the twins are together.

The relations among the coordinates in these three systems, called Lorentz transformations, are

$$x = \gamma(x' + vt') \qquad x' = \gamma(x - vt) \qquad x'' = \gamma(x + vt)$$
$$t = \gamma(t' + \overline{v}x') \qquad t' = \gamma(t - \overline{v}x) \qquad t'' = \gamma(t + \overline{v}x)$$

Here we are using $\overline{v} = v / c^2$, and $\gamma = 1/\sqrt{1-(v/c)^2}$ is the Lorentz factor, which gives the dilation/contraction ratio. (Because of the units we are using, $\overline{v}$ will have always the same numerical value as v, and we have chosen $v > 0$, even for the westbound frame S''.***

We are going to increase the speed of Beth's ship to $v = 0.96c$. This will enhance the effects of the speed and is handy because of the numerical coincidence that $1 - 0.96^2 = \left(\frac{28}{100}\right)^2$, so that now not only $\gamma = \frac{100}{28}$ is rational, but $\frac{v}{c} = \frac{96}{100}$ is also. (Formerly v/c was $\sqrt{3}/2 \cong 0.866$, and γ of course was 2.)

This time we will account for Beth's startup, but we are going to use infinite acceleration to do it. This un-physical idealization will be removed in the next section, where we will consider a gradual increase in speed. For now, we say that at $t' = 0$, Beth's ship suddenly begins traveling at $v = 0.96c$ to the east.

Figure 5 shows a sequence of graphs for the journey from the perspective of Art on Earth. There is a lot of information given, and key parts are marked with the colored vertical bars. Each of the five graphs (a) through (e) is a different time t in the Earth frame S. Note that in the S pictures, the green axis is just the x axis at that particular time t. For the other frames positions and times are mixed so that the interpretation is less straightforward, but in each case the pair of numbers above and below an axis give the spacetime coordinates of a single event, shown vertically, in each of the three frames.

In Figure 5(a), at the start of the trip, we can imagine Beth in her ship jumping from S to S', and thus taking up the speed v. Figure 5(b) then shows the situation when Art's clock has reached the 7-year mark. The ship is at $x =$ 6.72 light-years from Earth. The $x' = 0$ in S' denotes Beth's position, as we said, and we see from t' that she has aged 1.96 years.

In the spirit of Section 2, a single graph from Beth's perspective is given in Figure 6, for $t' =$ 7 years. We see that from Beth's point of view the Earth lies 6.72 light-years west of the ship, and she has aged 7 years while Art has aged 1.96 years.

Returning to Art, in Figure 5(c), 25 years have passed on Earth and only 7 years on the ship, which is now 24 light-years from home. At this point Beth decides to reverse her direction, and she instantly accelerates from v to $-v$. We can imagine her in her ship jumping now from S' to S'', where the x'' coordinate is 171.4 light-years and the clock reads $t'' =$ 171.6 years. These large values are due to our choice $x'' = x = 0$ at $t'' = t = 0$, which simplifies the math, and are not meaningful of themselves as far as the twins are concerned. Their significance is that $x'' =$ 171.4 light-years is the new tag to identify Beth's position, and that $t'' =$ 171.6 years marks, from S', Beth's having aged 7 years.

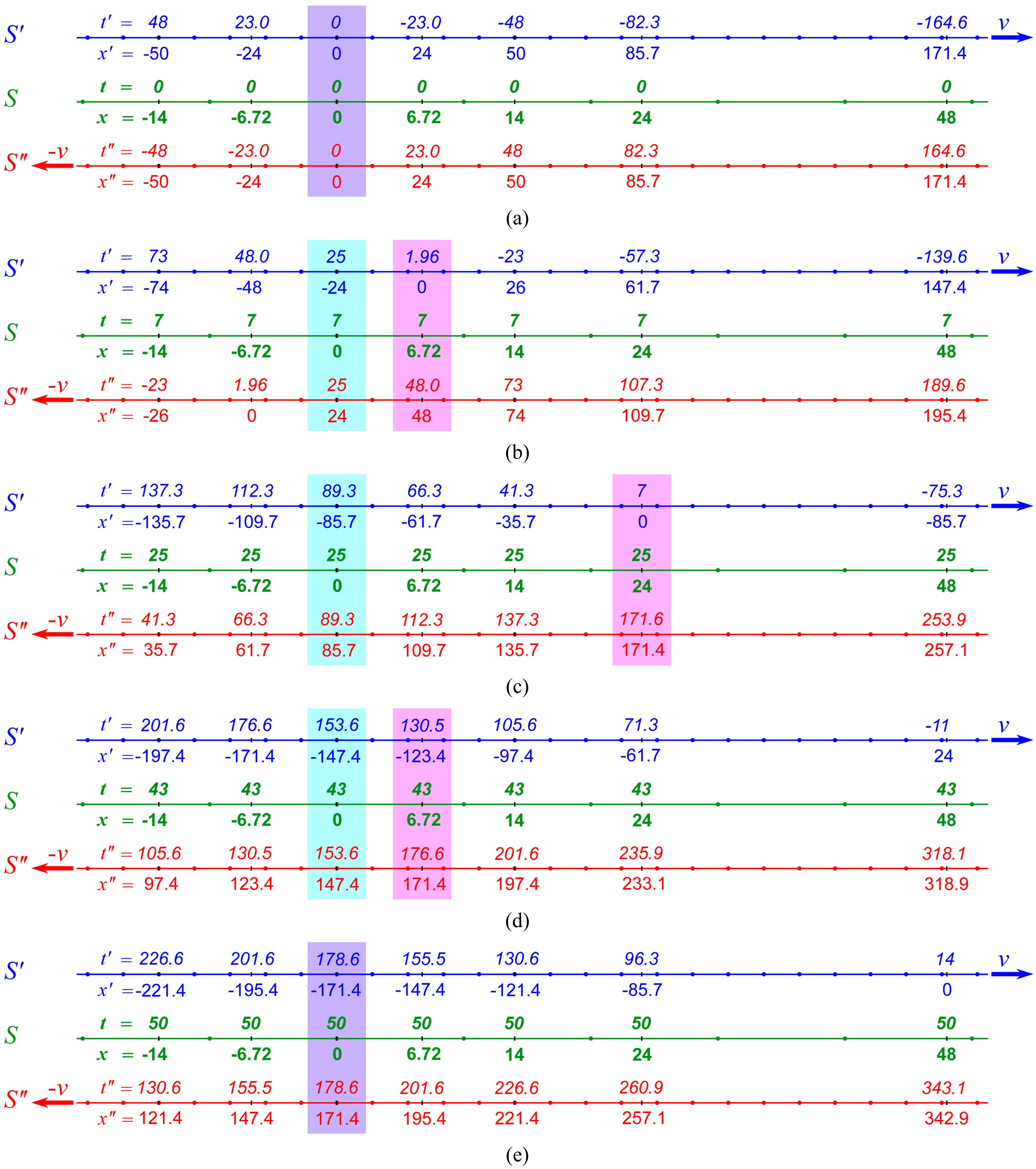

Figure 5. Frames of reference for the twins' journey, all seen from Art's perspective on Earth. Top to bottom, (a) to (e), five times t in the Earth frame S are shown along with the eastbound S' and the westbound S''. The position of Earth is marked in cyan, that of the ship in magenta, and the purple bar is when the two are together. The dots on the axes are spaced at 10 light-year intervals.

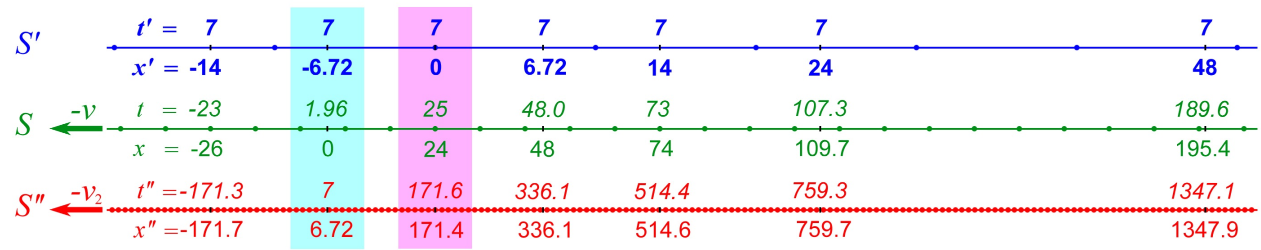

Figure 6. A graph from Beth's perspective, showing the reciprocity of relative motion. Compare with Figure 5(b). The combined velocity for S'' is $v_2 = 2v/(1+v^2/c^2)$, which is 0.99917 c.

Then in Figure 5(d), where Art is 43 years older, Beth has aged an additional 176.61 − 171.57 = 5.04 years. Finally in 5(e), after aging another 1.96 years, Beth comes suddenly to a stop, hopping back to S. She is 14 years older than when she left, and Art is 50 years older. Again, it is the change of speed, the change from one frame to another, which results in Beth, and not Art, having aged less. Art remains in S throughout, while Beth moves from S to S' to S'' and back to S.

We note in closing that the Lorentz equations provide a handy rule for the time between moving clocks. For a fixed time t in S, for example, two moving clocks (in S') a distance x apart will show times differing by $\Delta t' = \gamma \bar{v}\, x$. What this amounts to is that if you observe two co-moving clocks at different places, the then time difference on the clocks is just the light travel time for the distance between them *in the moving frame* ($\gamma x / c$) multiplied by the relative velocity factor v/c.

In Figure 5(c), the ship and the Earth are separated by $x = 24$ light-years, and the S' clocks differ by $\dfrac{\gamma x}{c}\dfrac{v}{c} = \dfrac{100}{28} 24\,\text{years} \times 0.96 = 82.3$ years, as shown in the graph. Similarly, in Figure 1, bottom picture, the ship is $\sqrt{3/4}$ light-seconds from earth, v/c is $\sqrt{3/4}$ and γ is 2, so the time difference between the ship and drone W2 clocks is $\Delta t' = 2 \times 3/4 =$ 1.5 seconds. In all cases, the clock further upstream has the earlier time, and the clock further downstream the later.

5. The real journey

In order to determine the details of a realistic trip, we need to know how things behave when they are accelerating. The frames of reference that we have been using are inertial frames, meaning they only move with constant speed relative to each other (and in a straight line). We should also mention that by always starting things off with the clocks synchronized at the origin we are using these frames in what is called "standard configuration." It is to such frames that the Lorentz transformations as given in Section 4 apply.

Now we know that an object stationary in S', like Beth, moves at a velocity v in S. But what if the object is not stationary? Let us use u instead of v for the relative velocity of S and S', and save v for $\dfrac{dx}{dt}$, which is generally different from u for moving objects. From the Lorentz equations for x and t, we see that $\dfrac{dx}{dt} = \dfrac{dx/dt'}{dt/dt'} = \dfrac{u+v'}{1+uv'/c^2}$, where $v' = \dfrac{dx'}{dt'}$ is the speed of the object in S'. When

$v' = 0$, as with Beth, then $u = v$ as we would expect. In this case, we also get $dx/dt' = \gamma u$ and $dt/dt' = \gamma$, so that $dx = dt'\ v/\sqrt{1 - v^2/c^2}$ and $dt = dt'/\sqrt{1 - v^2/c^2}$.

We can carry this another step and get $\dfrac{dv}{dt'} = \dfrac{1}{1 + uv'/c^2}\dfrac{dv'}{dt'} - \dfrac{u + v'}{(1 + uv'/c^2)^2}\dfrac{u}{c^2}\dfrac{dv'}{dt'}$ $= \dfrac{dv'}{dt'}\dfrac{1 - u^2/c^2}{(1 + uv'/c^2)^2}$. Dividing this by $\dfrac{dt}{dt'} = \dfrac{1 + uv'/c^2}{\sqrt{1 - u^2/c^2}}$ gives the acceleration dv/dt of the object in S in terms of its acceleration dv'/dt' in S', but if we simply multiply it by dt' we get the relation between infinitesimal changes in velocity, $dv = dv'\dfrac{1 - u^2/c^2}{(1 + uv'/c^2)^2}$, which is sufficient to our purposes. Again, if $v' = 0$, then $dv = dv'(1 - u^2/c^2)$, or better, $dv' = dv/(1 - v^2/c^2)$.

Our interest here is in an object which moves in S with a variable velocity $v(t)$, always moving in the x direction but changing speed with the time t. At any particular instant t, we place the object at the origin of an inertial frame S' moving with velocity $u = v(t)$ with respect to S. Its velocity in S', v', is zero, but the object is subject to an acceleration α which results in an infinitesimal change in velocity $dv' = \alpha dt'$. In S, this is $\alpha dt' = dv/(1 - v^2/c^2)$.

In the next instant, $t + dt$, we will need a new inertial frame with velocity $v(t) + dv$. And then another, and so on. Over a finite stretch of time from t_0 to t in S (or t'_0 to t' in S'—for $x' = 0$, $t \propto t'$), the velocity will change from v_0 to v, with $\int_{v_0}^{v} dv'/(1 - v'^2/c^2) = \int_{\tau_0}^{\tau} \alpha(\tau')\, d\tau'$. Here we have used τ instead of t' to signify that we are no longer dealing with a single inertial frame, but rather a continuous sequence of instantaneous frames (the tic mark ' indicates dummy variables for integration). Applying the same reasoning to the x and t differentials from above gives us $x - x_0 = \int_{x_0}^{x} dx =$ $\int_{\tau_0}^{\tau} d\tau'\ v/\sqrt{1 - v^2/c^2}$ and $t - t_0 = \int_{t_0}^{t} dt = \int_{\tau_0}^{\tau} d\tau'/\sqrt{1 - v^2/c^2}$. Note that in this formulation u and $v = dx/dt$ are identical, since the object is at rest in each instantaneous frame S'.

This quantity τ is called the proper time of the object. It is the time kept by a clock moving with the object, and it determines how the object ages. In cases where the velocity is constant τ is identical with a single t', as with Beth, for example. In the preceding section t' (and then later t'') was Beth's proper time.

The acceleration function $\alpha(\tau)$ gives the acceleration that would be measured by an instrument traveling with the object, and the integral $\int_{\tau_0}^{\tau} \alpha(\tau')\, d\tau'$ is a total (specific) impulse for the time interval. For our purposes a constant acceleration $\alpha(\tau) = a$ is fine, and for this case its integral is simply $\int_{\tau_0}^{\tau} \alpha(\tau')\, d\tau' = a(\tau - \tau_0)$.

The x and t integrals may be evaluated from the v integral, which is $\int_{v_0}^{v} dv'/(1-v'^2/c^2) =$ $c\left(\operatorname{atanh}\frac{v(\tau)}{c} - \operatorname{atanh}\frac{v(\tau_0)}{c}\right) = a(\tau-\tau_0)$, where atanh is the inverse function for the hyperbolic tangent tanh, satisfying $z = \tanh(\operatorname{atanh} z)$. This gives $\frac{v(\tau)}{c} = \tanh f(\tau)$, with $f(\tau) = \frac{a}{c}(\tau-\tau_0) + \operatorname{atanh}\frac{v(\tau_0)}{c}$, so that $x - x_0 = \int_{\tau_0}^{\tau} d\tau'\, v/\sqrt{1-v^2/c^2} = \int_{\tau_0}^{\tau} d\tau'\, c\tanh f(\tau')/\sqrt{1-\tanh^2 f(\tau')} = c\int_{\tau_0}^{\tau} d\tau' \sinh f(\tau')$ $= (c^2/a)[\cosh f(\tau) - \cosh f(\tau_0)]$. Similarly, $t - t_0 = \int_{\tau_0}^{\tau} d\tau'/\sqrt{1-\tanh^2 f(\tau')} = \int_{\tau_0}^{\tau} d\tau' \cosh f(\tau')$ $= (c/a)[\sinh f(\tau) - \sinh f(\tau_0)]$.

This, then, gives us what we need to proceed with accelerations in our space flight. To simplify the notation, we make the following definitions (the first symbol is "nu"):

$$\nu = a/c \qquad \tau_V = (1/\nu)\operatorname{atanh}[v(\tau_0)/c] \qquad \tau_C = \tau_0 - \tau_V .$$

Then the equations for x, t and v are

$$x(\tau) = x_0 + (c/\nu)[\cosh \nu(\tau-\tau_C) - \cosh \nu\tau_V]$$

$$t(\tau) = t_0 + (1/\nu)[\sinh \nu(\tau-\tau_C) - \sinh \nu\tau_V]$$

$$v(\tau) = c\tanh \nu(\tau-\tau_C)$$

These equations apply to conditions of acceleration, $|a| > 0$. As a matter of convention, let us at this point define $a > 0$. For acceleration in the negative x direction, ν in these equations will be replaced with $-\nu$.

Beth's journey this time will be the same as before, except that we will substitute acceleration segments for the instant jumps between frames. Thus there will be four acceleration segments and two steady velocity, or "cruise," segments. The two cruise segments will be the same length and at the same speed, though in opposite directions, and each of the acceleration segments will have the same acceleration and length, though here again oppositely directed in pairs. Also, we will not limit ourselves to a single choice of cruising speed or duration, or of acceleration, but keep it general so that we may compare various choices. For $v = 0.96c$ and very large acceleration a, we would expect to reproduce the results of Section 4.

We will go through the trip in detail, but we can already define two elapsed times, τ_a for an acceleration segment and τ_b for a cruise segment. These are both in ship times, and they will have counterparts for Earth time. Also, we will use number subscripts, 1 - 6, to mark the end of each segment; τ_3 for example is the time at the end of the first deceleration, when Beth will turn around. Note that τ_0 is a special case. It is an initial value, like τ_V and τ_C, and will change from segment to segment. In practice, we will choose the acceleration a, the cruising speed v_b and the cruise duration τ_b, and everything else will follow.

We have set up our equations with τ as the independent variable, and we will construct the journey from Beth's perspective, setting the times by her clock. However, we will be able to quickly convert these times to Art's frame using the $t(\tau)$ equation. Moreover, Beth's position will be given in

Art's frame by default (ideally she is always at zero in her own frame, but as we have seen already, this need not be the case).

The cruising speed is v_b. This, given the acceleration a, fixes the duration of the acceleration segments to $\tau_a = (1/\nu)\ \mathrm{atanh}(v_b/c)$. This follows for the first segment directly from the $v(\tau)$ equation; since τ_0 and $v(\tau_0)$ are both zero, so are τ_V and τ_C. For the other acceleration segments, we know from the symmetry that they will take the same time, but we will check them anyway. At the end of the initial segment, the position of the ship is $x(\tau_a) = (c/\nu)(\cosh \nu\tau_a - 1)$, and the time on Earth is $t(\tau_a) = (1/\nu)\sinh \nu\tau_a$. Note that for v_b, $\gamma = \gamma_b = 1/\sqrt{1-\tanh^2 \nu\tau_a} = \cosh \nu\tau_a$.

From Beth's perspective, her clock reads τ_a, and if there is a station nearby keeping Earth time, it will show $t(\tau_a)$. The Earth is a distance $\delta x' = x(\tau_a)/\gamma_b$ west of the ship. The time she sees on Art's clock is $t = t(\tau_a) - \overline{v}_S\, x(\tau_a) \ = t(\tau_a)[1 - \overline{v}_S\, x(\tau_a)/t(\tau_a)] \ = t(\tau_a)\mathrm{sech}\, \nu\tau_a \ = t(\tau_a)/\gamma_b$.

We choose a cruise time of τ_b, and this fixes the length of those segments to $v_b\tau_b$ in Beth's frame. For Art, the time elapsed from $\tau_1 = \tau_a$ to $\tau > \tau_1$ is, from the Lorentz equations, $t(\tau) - t(\tau_a) =$ $\gamma_b(\tau + \overline{v}_b x') \ - \gamma_b(\tau_a + \overline{v}_b x') \ = \gamma_b(\tau - \tau_a)$, where x' is Beth's position. Similarly, $x(\tau) - x(\tau_a) =$ $\gamma_b(x' + v_b\tau) \ - \gamma_b(x' + v_b\tau_a) \ = \gamma_b v_b(\tau - \tau_a)$. Thus Beth's time at the end of the eastbound cruise is $\tau_2 = \tau_a + \tau_b$. Art's clock now reads $t(\tau_2) = \ t(\tau_a) + \gamma_b\tau_b$, and his position for the ship is $x(\tau_2) =$ $x(\tau_a) + \gamma_b v_b \tau_b$.

The initial conditions then for the eastbound deceleration are $x_0 = x(\tau_2)$, $t_0 = t(\tau_2)$, $\tau_0 = \tau_2$ and $v(\tau_0) = v_b$; then, since now the acceleration is negative, $\tau_V = (-1/\nu)\ \mathrm{atanh}(v_b/c) \ = -\tau_a$, and $\tau_C = \tau_0 - \tau_V \ = 2\tau_a + \tau_b$. The final condition is $v(\tau_3) = 0 \ = -c\tanh \nu(\tau - \tau_C)$, so $\tau_3 = \tau_C$ is the one-way trip time.

The ship's position during this segment is $x(\tau) = \ x_0 +$ $(-c/\nu)[\cosh(-\nu(\tau - \tau_C)) - \cosh(-\nu(-\tau_a))] \ = x(\tau_2) - (c/\nu)[\cosh \nu(\tau - \tau_3) - \cosh \nu\tau_a]$.

The Earth time is $t(\tau) = t_0 + \ (-1/\nu)[\sinh(-\nu(\tau - \tau_C)) - \sinh(-\nu(-\tau_a))] \ = t(\tau_2) +$ $(1/\nu)[\sinh \nu(\tau - \tau_3) + \sinh \nu\tau_a]$.

At the stop, $x(\tau_3) = \ x(\tau_2) - \ (c/\nu)(1 - \cosh \nu\tau_a) \ = 2x(\tau_a) + \gamma_b v_b \tau_b$, and $t(\tau_3) = t(\tau_2) + \ (1/\nu)\ \sinh \nu\tau_a \ = 2t(\tau_a) + \gamma_b\tau_b$.

Since Beth is turning around and heading back immediately, the eastbound deceleration leg actually continues into the westbound acceleration leg. Now we are looking for $v(\tau) \to -v_b$ from the same initial conditions, that is $v(\tau_4) = -v_b = -c\tanh \nu(\tau_4 - \tau_3)$, so that $\tau_4 = \tau_3 + (1/\nu)\mathrm{atanh}\, v_b/c$ $= 3\tau_a + \tau_b$. The position is $x(\tau) = \ x(\tau_2) - \ (c/\nu)[\cosh \nu(\tau - \tau_3) - \cosh \nu\tau_a]$, which finishes, since

$\tau_4 - \tau_3 = \tau_a$, at $x(\tau_4) = x(\tau_2)$. The time is $t(\tau_2) +\ (1/\nu)[\sinh\nu(\tau-\tau_3) + \sinh\nu\tau_a]$, which gives $t(\tau_4) = t(\tau_2) + 2t(\tau_a)\ = 3t(\tau_a) + \gamma_b\tau_b$.

For the westbound cruise segment, $v = -v_b$ and the Lorentz equations give $t(\tau) - t(\tau_4) =$ $\gamma_b(\tau - \bar{v}_b x')\ - \gamma_b(\tau_4 - \bar{v}_b x')\ = \gamma_b(\tau - \tau_4)$ and $x(\tau) - x(\tau_4) =\ \gamma_b(x' - v_b\tau)\ - \gamma_b(x' - v_b\tau_4)\ = -\gamma_b v_b(\tau - \tau_4)$. We get $\tau_5 = \tau_4 + \tau_b\ = 3\tau_a + 2\tau_b$, $t(\tau_5) =\ t(\tau_4) + \gamma_b\tau_b\ = 3t(\tau_a) + 2\gamma_b\tau_b$ and $x(\tau_5) =\ x(\tau_4) - \gamma_b v_b\tau_b$ $= x(\tau_a)$.

For the final segment we again have positive acceleration, and the initial conditions are $x_0 = x(\tau_a)$, $t_0 = t(\tau_5)$, $\tau_0 = \tau_5$, $v(\tau_0) = -v_b$, $\tau_V = (1/\nu)$ atanh$(-v_b/c)\ = -\tau_a$ and $\tau_C = \tau_0 - \tau_V$ $= 4\tau_a + 2\tau_b$. The final condition is $v(\tau_6) = 0$, where $\tau_6 = \tau_C$ is the round trip time. The ship's position is $x(\tau) =\ x_0 +\ (c/\nu)[\cosh(\nu(\tau-\tau_C)) - \cosh(\nu(-\tau_a))]\ = x_0 +\ (c/\nu)[\cosh\nu(\tau-\tau_6) - \cosh\nu\tau_a]$, and this goes to $x(\tau_a) - x(\tau_a) = 0$. The time is $t(\tau) = t_0 +\ (1/\nu)[\sinh\nu(\tau-\tau_6) + \sinh\nu\tau_a]$, which finishes at $t(\tau_6) = t(x_5) + t(\tau_a)\ = 4t(\tau_a) + 2\gamma_b\tau_b$.

Table 1 summarizes the results for our journey. To be clear, τ is the time on Beth's clock on the ship, and $x(\tau)$ and $t(\tau)$ are the position and time in the Earth frame for the ship at this time τ, while $v(\tau)$ is the relative velocity of the two frames, that is, of the ship relative to Earth. Elsewhere the time observed in the Earth frame on a clock moving with the ship is different from τ by $\gamma\bar{v}\,\delta x$, where δx is the distance of the clock from $x(\tau)$ (in S). The position of the Earth in the ship frame is $-x(\tau)/\gamma(\tau)$.

Now let's look at some numbers. Before we get into practical cases, we will check our results against those of Section 4. With $v_b = 0.96c$ and $\tau_b =$ 7 years, choose a so that $\tau_b >> \tau_a$. For example, say $\tau_b/\tau_a = 10^6$, which gives $\tau_a = 7\times10^{-6}$ years. Then $\nu = (1/\tau_a)$ atanh$(v_b/c) \rightarrow$ $\nu = 0.00881$ s^{-1} and $a = 2{,}640{,}835$ m s^{-2}. So we have a trip time of $\tau_6 = 4\tau_a + 2\tau_b =$ 14.000028 years for Beth, and for Art $t(\tau_6) =\ (4/\nu)\sinh\nu\tau_a\ + 2\gamma_b\tau_b =$ 50.000049 years.

A standard choice for space voyages is $a = g$, the 1 gee of gravity on the Earth (at sea level), which is 9.80665 m s^{-2}. Let's consider our original journey, $v_b = 0.96c$ and $\tau_b =$ 7 years, with this acceleration. We find $\tau_a = 1.885$ years and $t(\tau_a) =$ 3.321 years. That is, during the initial acceleration segment, Beth ages 1.885 years and Art ages 3.321 years.

Figure 7 shows a graph of the time on Beth's clock τ, the position of the ship (in Art's frame) x and the relative velocity v, plotted against the time on Art's clock t. In Figure 8 are plotted the distance from Earth and the time on Art's clock as seen by Beth ("seen" in the sense of instant observation), and the time on drone clocks passing Earth as seen by Art (in real time). The drones are a carry-over from Section 2. If we imagine a string of them all moving in tandem with the ship but strung out to an indefinite distance to her west, then these clocks, as they pass the Earth, show (during cruise segments) the double-rate we talked about in Section 2 ("outside the window").

Table 1. Times, positions and velocities for the round-trip.

Ship time τ (Ship Frame)	Earth time t (Earth Frame)	Ship position x (Earth frame)	Velocity v
$0 \le \tau \le \tau_1$ $\tau_1 = \tau_a = \frac{1}{\nu}\operatorname{atanh}\frac{v_b}{c}$	$\frac{1}{\nu}\sinh \nu\tau$	$\frac{c}{\nu}(\cosh \nu\tau - 1)$	$c \tanh \nu\tau$
$\tau_1 \le \tau \le \tau_2$ $\tau_2 = \tau_a + \tau_b$	$t(\tau_a) + \gamma_b(\tau - \tau_a)$	$x(\tau_a) + \gamma_b v_b(\tau - \tau_a)$	$v_b = c \tanh \nu\tau_a$
$\tau_2 \le \tau \le \tau_3$ $\tau_3 = 2\tau_a + \tau_b$	$2t(\tau_a) + \gamma_b\tau_b +$ $\frac{1}{\nu}\sinh \nu(\tau - \tau_3)$	$2x(\tau_a) + \gamma_b v_b \tau_b -$ $\frac{c}{\nu}[\cosh \nu(\tau - \tau_3) - 1]$	$-c \tanh \nu(\tau - \tau_3)$
$\tau_3 \le \tau \le \tau_4$ $\tau_4 = 3\tau_a + \tau_b$	same as $\tau_2 \le \tau \le \tau_3$	same as $\tau_2 \le \tau \le \tau_3$	same as $\tau_2 \le \tau \le \tau_3$
$\tau_4 \le \tau \le \tau_5$ $\tau_5 = 3\tau_a + 2\tau_b$	$3t(\tau_a) + \gamma_b(\tau - 3\tau_a)$	$x(\tau_a) - \gamma_b v_b(\tau - \tau_5)$	$-v_b = -c \tanh \nu\tau_a$
$\tau_5 \le \tau \le \tau_6$ $\tau_6 = 4\tau_a + 2\tau_b$	$4t(\tau_a) + 2\gamma_b\tau_b +$ $\frac{1}{\nu}\sinh \nu(\tau - \tau_6)$	$\frac{c}{\nu}[\cosh \nu(\tau - \tau_6) - 1]$	$c \tanh \nu(\tau - \tau_6)$

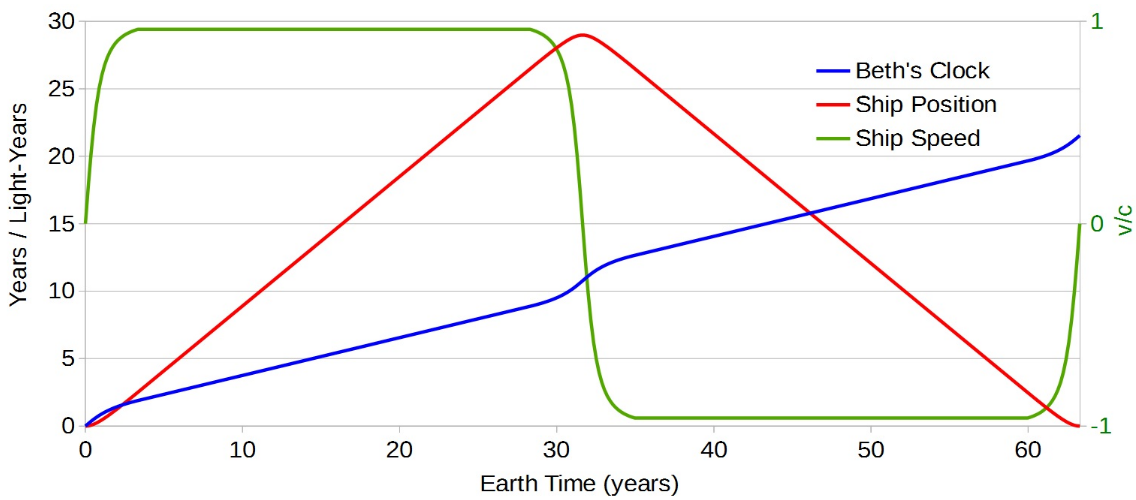

Figure 7. Journey of Section 4 with 1-gee acceleration.

The trip time is 63.3 years, Art’s age increase, and Beth comes home 21.5 years older, having traveled 29 light-years and back, for a total distance of 58 light-years.

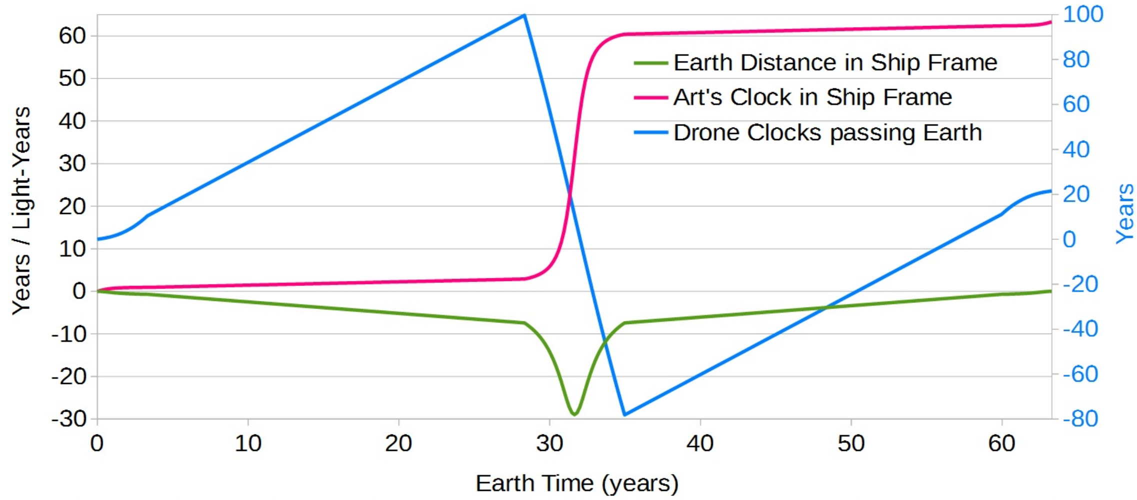

Figure 8. Further details of Figure 7.

It is interesting to look at the effect of cruising speed v_b on the mission outcome. We have done this in Figure 9 by keeping $a = g$ and $\tau_b =$ 7 years, and varying v_b from 0.5 to 0.98 c. The trip time for the ship goes from 16 to 23 years while the Earth time starts at a little over 18 years and climbs steeply to over 89 years. The round-trip distance varies from 8.7 to 84.5 light-years.

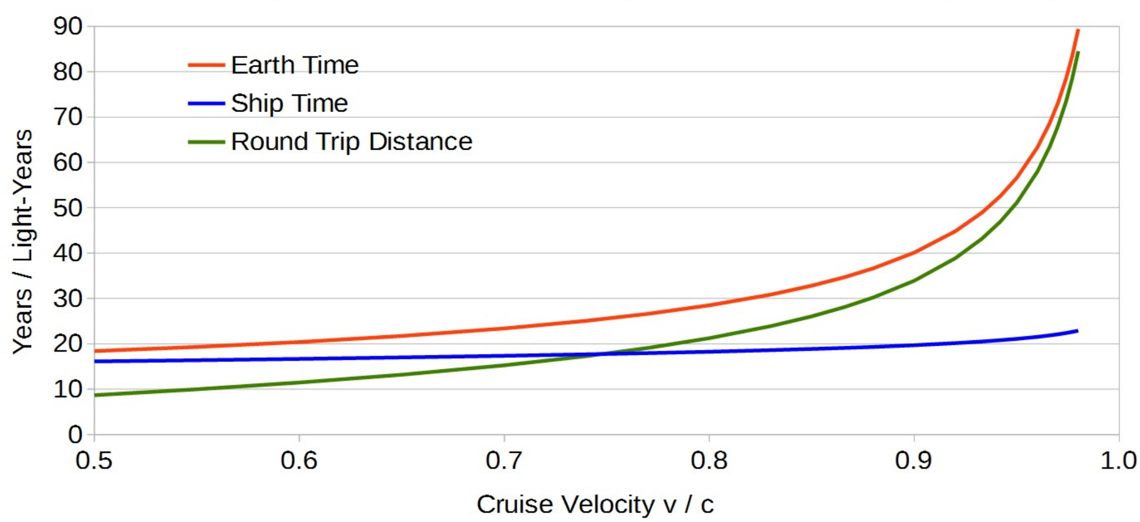

Figure 9. Times and distances for increasing cruise speed, with $a = g$ and a total cruise time of $2\tau_b =$ 14 years.

The effect of acceleration for $v_b = 0.96c$ and $\tau_b =$ 3 years is shown in Figure 10. The greater distances and times for small accelerations are due to the longer periods required to attain cruising speed. For $a > 1$, changes in τ_a become less important compared with the fixed τ_b.

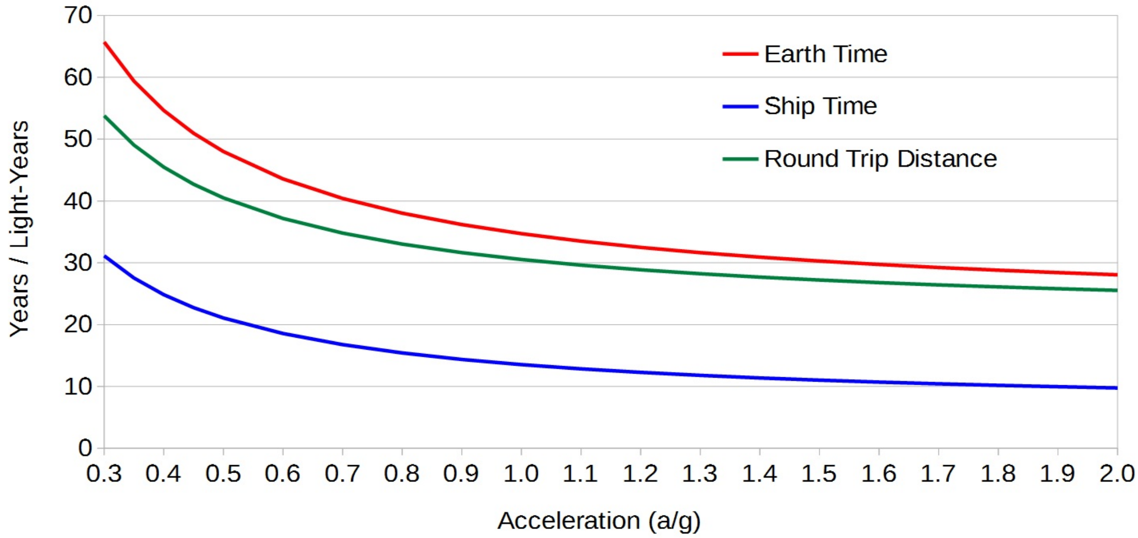

Figure 10. Mission outcomes for $v_b = 0.96\,c$ and $\tau_b =$ 3 years with variable acceleration.

We conclude with a few key takeaways from the full calculation here. First, note that in Figure 8, the curve showing Art's clock in the ship frame bears out what we saw in Beth's frame jump (from S' to S'') in Section 4. It is the change in speed that decides the age race in Beth's favor, as it were. From the beginning of her eastbound deceleration to the beginning of her westbound cruise, she sees Art's clock go from 6 years behind hers to 48 years ahead (compare Beth's clock from Figure 7).

Also interesting, from Figure 7, is that the effect of acceleration on proper time is the same for both directions. When Beth reverses direction in the middle of the graph, the slope $d\tau / dt$ is continuous. The function $d\tau / dt = 1/\cosh \nu\tau$ is an "even" function, the same for $\pm\nu$. Moreover, $d\tau / dt$ is always less than 1, so that acceleration invariably slows aging. This is in contrast to v_b which, so long as it lasts, affects both twins equally. Unlike velocity, acceleration is asymmetric. Beth feels the force of the acceleration, whereas motion at constant velocity, for Art and Beth alike, is undetectable of itself.

As a footnote, we point out that while the trips depicted in this section are physically realistic, they are, so far at least (2021), technologically beyond our ability. There is no known means of sustaining gee-type accelerations of macroscopic payloads for days on end, much less months. A practical limit with current hardware would be on the order of an hour. Looking ahead to improved propulsion techniques, it is also worth noting that there are other challenges, such as extended periods of hard radiation, which is vastly exacerbated at high speeds. Nonetheless, such journeys are fully in accord with the laws of physics and may one day take place.